Introduction to Logic - Exam Jan. 26, 2016 - Model answers

A1. a) A : Alice will be fired
G : The policeman is grumpy
T : The policeman is tired
D : Alice is drunk

$$\neg A \lor (G \land T) \lor D$$

b) L : Alice likes the book
D : Deepika likes the book
B : I will buy the book

$$(\neg L \land \neg D) \rightarrow \neg B$$

Also correct: $\neg(L \lor D) \rightarrow \neg B$

A2 a : Trump
b : Sanders
c : Jason
d : Clinton

$D(x)$: x is a democrat
$S(x,y)$: x shouts louder than y
$H(x,y)$: x hates y
$P(x,y,z)$: x prefers y to z

a) $\forall x (x \neq a \rightarrow H(a,x)) \rightarrow \forall y (y \neq a \rightarrow H(y,a))$. Also OK:
$\ldots \leftrightarrow \ldots \leftrightarrow$

b) $\neg \exists x (D(x) \land S(x,b)) \land P(c,b,d)$

3a)

1 $(A \lor \neg B) \to C$

2 A

3 $A \lor \neg B$ — $\lor$ Intro: 2

4 C — $\to$ Elim: 1,3

5 $A \to C$ — $\to$ Intro: 2-4

6 $\neg C$

7 $\neg B$

8 $A \lor \neg B$ — $\lor$ Intro: 7

9 C — $\to$ Elim: 1, 8

10 $\bot$ — $\bot$ Intro: 9, 6

11. $\neg\neg B$ — $\neg$ Intro: 7-10

12. B — $\neg$ Elim: 11

13. $\neg C \to B$ — $\to$ Intro: 6-12

14. $(A \to C) \land (\neg C \to B)$ — $\land$ Intro: 5, 13

3b)

1 $\neg B \lor B$

2 $\neg B$

3 $(B \to A) \to B$

4 B

5 $\bot$ — $\bot$ Intro: 4, 2

6 A — $\bot$ Elim: 5

7 $B \to A$ — $\to$ Intro: 4-6

8 B — $\to$ Elim: 3,7

9 $((B \to A) \to B) \to B$ — $\to$ Intro: 3-8

10 B

11 $(B \to A) \to B$

12 B — Reit: 10

13 $((B \to A) \to B) \to B$ — $\to$ Intro: 11-12

14 $((B \to A) \to B) \to B$ — $\lor$ Elim: 1, 2-9, 10-13

3b) A shorter proof, without need of $B \vee \neg B$

1	$\neg B \vee B$	
2	$(B \to A) \to B$	
3	$\neg B$	
4	B	
5	$\perp$	$\perp$ Intro: 4, 3
6	A	$\perp$ Elim: 5
7	$B \to A$	$\to$ Intro: 4-6
8	B	$\to$ Elim: 2, 7
9	$\perp$	$\perp$ Intro: 8, 3
10	$\neg\neg B$	$\neg$ Intro: 3-9
11	B	$\neg$ Elim: 10
12	$((B \to A) \to B) \to B$	$\to$ Intro: 2-11

3d) A shorter proof, without need of $\exists$-Elim:

1.	$a = b$	
2.	$b = c$	
3.	$a = c$	$=$ Elim: 1, 2
4	$a = a$	$=$ Intro
5	$c = a$	$=$ Elim: 4, 5
6	$\exists x\, R(x, c)$	
7	$\exists x\, R(x, a)$	$=$ Elim: 6, 5
8	$\exists x\, R(x, c) \to \exists x\, R(x, a)$	

3 c)

1 $\forall x \forall y \forall z ((R(x,y) \land R(y,z)) \to \neg R(x,z))$

2 $\exists x\, R(x,x)$

3 $\boxed{a}$, $R(a,a)$

4. $\forall y \forall z ((R(a,y) \land R(y,z)) \to \neg R(a,z))$ — $\forall$ Elim: 1

5. $\forall z ((R(a,a) \land R(a,z)) \to \neg R(a,z))$ — $\forall$ Elim: 4

6. $(R(a,a) \land R(a,a)) \to \neg R(a,a)$ — $\forall$ Elim: 5

7. $R(a,a) \land R(a,a)$ — $\land$ Intro: 3, 3

8 $\neg R(a,a)$ — $\to$ Elim: 6,7

9 $\perp$ — $\perp$ Intro: 3, 8

10 $\perp$ — $\exists$ Elim: 2, 3-9

11 $\neg \exists x\, R(x,x)$ — $\neg$ Intro: 2-10

2

3 d)

(a longer proof, applying = Elim only to atoms

1 $a=b$

2 $b=c$

3. $a=c$ — = Elim: 1,2

4. $a=a$ — = Intro

5 $c=a$ — = Elim: 4,5

6 $\exists x\, R(x,c)$

7 $\boxed{d}$ $R(d,c)$

8 $R(d,a)$ — = Elim: 7, 5

9 $\exists x\, R(x,a)$ — $\exists$ Intro: 8

10 $\exists x\, R(x,a)$ — $\exists$ Elim: 6, 7-9

11 $\exists x\, R(x,c) \to \exists x\, R(x,a)$ — $\to$ Intro: 6-10

A4 a)

	P	Q	R	$P \leftrightarrow$	$(Q \vee R)$	$(P \to Q)$	$\vee$	$(\neg P$	$\to R)$
1	T	T	T	T	T	T	T	F	T
2	T	T	F	T	T	T	T	F	T
3	T	F	T	T	T	F	T	F	T
4	T	F	F	F	F	F	T	F	T
5	F	T	T	F	T	T	T	T	T
6	F	T	F	F	T	T	T	T	F
7	F	F	T	F	T	T	T	T	T
8	F	F	F	T	F	T	T	T	F

The conclusion is a tautological consequence of the premise because for all situations in which the premise is true (namely row 1, 2, 3, 8), the conclusion is true as well

A4 b.

	Large(a)	Larger(a,b)	Larger(b,a)	Large(a) → Larger(b,a)	Larger(a,b) → ¬Large(a)
spurious → 1.	T	T	T	T	FF
2.	T	T	F	F	FF
(TT-impossible) 3.	T	F	T	T	TF
4.	T	F	F	F	TF
spurious → 5.	F	T	T	T	TT
6.	F	T	F	T	TT
7.	F	F	T	T	TT
8.	F	F	F	T	TT

Rows 1 and 5 are spurious, because Larger(a,b) and Larger(b,a) cannot be true at the same time.

In all non-spurious rows in which Large(a) → Larger(b,a) is True (namely, rows 3, 6, 7, 8), the conclusion Larger(a,b) → ¬Large(a) is also true. So the conclusion is a logical consequence of the premise.

Remark: Row 3 is TT-impossible but not logically impossible, e.g. ~~b~~ could be 'Extra-Large'. However, we will not count it wrong if you mark row 3 as spurious

	A	B	C	$A \leftrightarrow (B \leftrightarrow C)$		$(A \leftrightarrow B) \leftrightarrow C$	
A4c. 1.	T	T	T	T	T	T	T
2.	T	T	F	F	F	T	F
3.	T	F	T	F	F	F	F
4.	T	F	F	T	T	F	T
5.	F	T	T	F	T	F	F
6.	F	T	F	T	F	F	T
7.	F	F	T	T	F	T	T
8	F	F	F	F	T	T	F
				↑			↑

The two formulas get the same truth values in all 8 rows, so they are tautologically equivalent

A5a. $\exists x \exists y (x \neq y \wedge Cube(x) \wedge Cube(y) \wedge SameColumn(x,y))$

b. i) F (*)
ii) T
iii) T
IV) T
V) T
VI) T
VII) T

c. If you remove c, then for objects a, e it holds that they are not cubes, but still there is no Cube at the back of them
[mentioning one of a, e suffices when giving a counterexample]

In A5 b i), we have also counted T correct, because according to some (older) versions of the definition, Between(d, e, b) was true

B1

$$(A \vee B) \Leftrightarrow (\neg B \rightarrow C) \overset{\text{def.} \Leftrightarrow}{\Longleftrightarrow}$$

$$[\neg(A \vee B) \vee (\neg B \rightarrow C)] \wedge (\neg(\neg B \rightarrow C) \vee (A \vee B)) \overset{\text{def} \rightarrow,\ 2x}{\Longleftrightarrow}$$

$$[\neg(A \vee B) \vee (\neg\neg B \vee C)] \wedge ((\neg B \wedge \neg C) \vee (A \vee B)) \overset{\text{dubble } \neg\neg \text{ + deMorgan}}{\Longleftrightarrow}$$

$$[(\neg A \wedge \neg B) \vee (B \vee C)] \wedge ((\neg B \wedge \neg C) \vee (A \vee B)) \overset{\text{distrib. law, 2x}}{\Longleftrightarrow}$$

$$[(\neg A \vee B \vee C) \wedge (\neg B \vee B \vee C)] \wedge ((\neg B \vee A \vee B) \wedge (\neg C \vee A \vee B))$$

This formula is already in CNF, but it can be further simplified (not obligatory):

$$\Leftrightarrow (\neg A \vee B \vee C) \wedge (A \vee B \vee \neg C)$$

B2a)

has to stay, because it is free

$$(A(x) \wedge \forall x \exists y\, R(x,y)) \rightarrow \exists y\, B(y) \overset{2x \text{ alf. variant}}{\Longleftrightarrow}$$

$$(A(x) \wedge \forall w \exists y\, R(w,y)) \rightarrow \exists z\, B(z) \Longleftrightarrow$$

$$\forall w\, (A(x) \wedge \exists y\, R(w,y)) \rightarrow \exists z\, B(z) \Longleftrightarrow$$

$$\forall w \exists y\, (A(x) \wedge R(w,y)) \rightarrow \exists z\, B(z) \Longleftrightarrow$$

$$\exists w \forall y\, [(A(x) \wedge R(w,y)) \rightarrow \exists z\, B(z)] \Longleftrightarrow$$

$$\exists w \forall y \exists z\, [(A(x) \wedge R(w,y)) \rightarrow B(z)]$$

This formula is in Prenex normal form.

b. The Skolem normal form for $\exists x \forall y \exists z \forall u \exists w (R(x,y,z) \rightarrow R(x,u,w))$ is: $\forall y \forall u (R(a,y,f(y)) \wedge R(a,u,g(y,u)))$

C.

A	B	C	D	E	A	∧ E	∧ (¬D	∨ ¬C	∨ ¬B	∨ A)	∧ (C	∨ ¬A)	∧ (D	∨ ¬E)
T	F	T	T	T	T	T	FT	FT	TF	T	T	FT	T	FT
2	4	3a	3a	2	2	2	3b 3a	3b 3a	4	2	3a	3a 2	3a	3a 2

We follow the steps of the algorithm, and in steps 2 and 3a, 3b, 3c we only make true those positive atoms that we are forced to make true.
In step 4, only one atom, B, is left, and we make this false. Indeed the resulting truth assignment makes the formula true.

B3. a. $\text{boss}(\text{boss}(c)) = \text{friend}(d)$

b. $\text{friend}(\text{boss}(a)) \neq d \wedge \text{friend}(\text{boss}(a)) \neq c \wedge$
$\text{Likes}(\text{boss}(a), d) \wedge \text{Likes}(\text{boss}(a), c)$

c. $\forall x\, (\forall y\, (x = \text{friend}(y)) \rightarrow \neg \exists z\, (x = \text{boss}(z)))$

B4. a) $\mathcal{M} \models P(a) \vee R(x,y)\, [h] \iff$
$\mathcal{M} \models P(a)\,[h]$ or $\mathcal{M} \models R(x,y)\,[h] \iff$
$[\![a]\!]^{\mathcal{M}}_h \in \mathcal{M}(P)$ or $\langle [\![x]\!]^{\mathcal{M}}_h, [\![y]\!]^{\mathcal{M}}_h \rangle \in \mathcal{M}(R) \iff$

$\mathcal{M}(a) \in \mathcal{M}(P)$ or $\langle h(x), h(y) \rangle \in \mathcal{M}(R) \iff$
$3 \in \mathcal{M}(P)$ or $\langle 2, 3 \rangle \in \mathcal{M}(R)$.
This is true because $\langle 2,3 \rangle \in \mathcal{M}(R)$.

b) $\mathcal{M} \models \exists x (R(x,x) \wedge \neg P(x))[h] \iff$
There is a $d \in \mathcal{M}(\forall)$ such that $\mathcal{M} \models R(x,x) \wedge \neg P(x)\, [h[{}^{x}/_{d}]] \iff$
There is a $d \in \mathcal{M}(\forall)$ s.t. $[\mathcal{M} \models R(x,x)[h[{}^{x}/_{d}]]$ and not $\mathcal{M} \models P(x)[h[{}^{x}/_{d}]]]$
$\iff$ There is a $d \in \mathcal{M}(\forall)$ s.t. $[\langle [\![x]\!]^{\mathcal{M}}_{h[x/d]}, [\![x]\!]^{\mathcal{M}}_{h[x/d]} \rangle \in \mathcal{M}(R)$ and

not $[\![x]\!]^{\mathcal{M}}_{h[x/d]} \in \mathcal{M}(P)$ $]$

$\iff$ There is a $d \in \mathcal{M}(\forall)$ such that $[\langle d, d \rangle \in \mathcal{M}(R)$ and not $d \in \mathcal{M}(P)]$

This is not true, because the only $d \in \mathcal{M}(\forall)$ with $\langle d,d \rangle \in \mathcal{M}(R)$ is $d = 2$, and we have $2 \in \mathcal{M}(P)$.

B4c) $\mathcal{M} \models \forall y \exists x R(y,x)[h] \iff$

for all $d \in \mathcal{M}(\forall)$, $\mathcal{M} \models \exists x R(y,x)[h[y/d]] \iff$

for all $d \in \mathcal{M}(\forall)$, there is an $e \in \mathcal{M}(\forall)$ s.t. $\mathcal{M} \models R(y,x)[h[y/d, x/e]] \iff$

$\langle [\![y]\!]^{\mathcal{M}}_{h[y/d, x/e]}, [\![x]\!]^{\mathcal{M}}_{h[y/d, x/e]} \rangle \in \mathcal{M}(R) \iff$

for all $d \in \mathcal{M}(\forall)$, there is an $e \in \mathcal{M}(\forall)$ such that $\langle d, e \rangle \in \mathcal{M}(R)$.

This is not true, because for $d = 3$, there is no $e \in \mathcal{M}(\forall)$ such that $\langle 3, e \rangle \in \mathcal{M}(R)$.

A6, Bonus

1	$\neg \forall x \exists y \neg A(x,y)$	
2	$\neg \exists x \forall y A(x,y)$	
3	$\boxed{c}$	
4	$\neg \exists y \neg A(c,y)$	
5	$\boxed{d}$	
6	$\neg A(c,d)$	
7	$\exists y \neg A(c,y)$	$\exists$ Intro: 6
8	$\bot$	$\bot$ Intro: 7, 4
9	$\neg\neg A(c,d)$	$\neg$ Intro: 6–8
10	$A(c,d)$	$\neg$ Elim: 9
11	$\forall y A(c,y)$	$\forall$ Intro: 5–10
12	$\exists x \forall y A(x,y)$	$\exists$ Intro: 11
13	$\bot$	$\bot$ Intro: 12, 3
14.	$\neg\neg \exists y \neg A(c,y)$	$\neg$ Intro: 4–13
15.	$\exists y \neg A(c,y)$	$\neg$ Elim: 14
16	$\forall x \exists y \neg A(x,y)$	$\forall$ Intro: 3–15
17	$\bot$	$\bot$ Intro: 1–16
18	$\neg\neg \exists x \forall y A(x,y)$	$\neg$ Intro: 2–17
19	$\exists x \forall y A(x,y)$	$\neg$ Elim: 18
20	$\boxed{a}$	
21	$\boxed{b}$ $\forall y A(b,y)$	
22	$A(b,a)$	$\forall$ Elim: 21
23	$\exists x A(x,a)$	$\exists$ Intro: 22
24	$\exists x A(x,a)$	$\exists$ Elim: 19, 21–23
25	$\forall y \exists x A(x,y)$	$\forall$ Intro: 20–24
26	$\exists x \forall y A(x,y) \wedge \forall y \exists x A(x,y)$	